Artificial Intelligence

Lecture 5 - Propositional Calculus I

Knowledge Representation & Reasoning

- Representation of information in a computer program
- Representations are *declarative* - it’s possible to give a precise account of what they *mean* which is independent of the operations performed on them
- Allows the same information to be used to solve different problems
- Reasoning is the application of general rules to information to derive new information, e.g., “*All men are mortal*”, “*Socrates is a man*”, therefore “*Socrates is mortal*”
- Reasoning procedure should be *truth preserving*, for some definition of ‘truth’

Outline

- Propositions and models
  - Valid inference
  - Propositions
  - Models
- Syntax
  - Logical connectives
  - Language
- Semantics
  - Truth tables
  - Logical equivalence
  - Satisfiability & validity
- Entailment
- Inference

Valid Inference

- Consider a simple example:
  - if the train is late and there are no taxis at the station, John is late for the meeting
  - the train is late
  - John is not late for the meeting
  - *Therefore* there were taxis at the station
- *Valid inference* is the derivation of *true statements* from other statements which are known (or assumed) to be true

Propositions

- The simplest kind of representations are declarative sentences
- The *content* of a declarative statement (what it is about) is called a *proposition*
- *Propositions* are statements about the world (or some domain of interest) that can be *true* or *false*
- For example, "*Paris is the capital of France*" is a proposition which is true if and only if Paris really is the capital of France
- The same proposition can be expressed by different sentences, in the same or different natural languages, e.g., "*snow is white*", "*schnee ist weiß*" . . .
- Only two (mutually exclusive) possibilities - truth values of proposition can't be "*both true and false*" or "*unknown*"

Models

- Definition (Model)
  - Given a set of atomic propositions $P = \{p, q, r, \ldots\}$ (propositional variables) describing a state of affairs or a problem to be solved, a *model* is an assignment of true or *false* to each atomic proposition p
- Example
  - For example, a particular model might assign *true* to p and *false* to q, while in another model both p and q are *false*
  - With n propositions, i.e., $n = |P|$, the number of possible models is 2^n

Logical Connectives

- We can form more complex declarative sentences using *logical connectives*:
  - $\neg$ negation "not"
  - $\wedge$ conjunction "and"
  - $\vee$ disjunction "or"
  - $\rightarrow$ conditional "implies"

Examples of Complex Sentences

- Given the atomic propositions:
  - p: I went to the cinema last week
  - q: I went to the theatre last week
  - r: I bought a ticket
- We can construct more complex sentences such as:
  - $\neg p$: I did **not** go to the cinema last week
  - $p \wedge q$: I went to the cinema **and** the theatre last week
  - $p \vee q$: I went to the cinema **or** the theatre last week
  - $p \rightarrow r$: **if** I went to the cinema last week **then** I bought a ticket

Language

- Definition (Formula)
  - Given a set P of propositional variables, the formulas of the propositional calculus are defined as:
    1. any $p \in P$ is a formula;
    2. if φ is a formula, then $\neg\varphi$ is a formula;
    3. if φ and ψ are formulas, then $\varphi \wedge \psi$, $\varphi \vee \psi$, and $\varphi \rightarrow \psi$ are formulas
- Precedence (and/or parentheses) are used to disambiguate complex sentences:
  - $\neg$ binds more tightly than $\wedge$ or $\vee$, which in turn bind more tightly than $\rightarrow$
- For example $\neg p \vee q \rightarrow r$ is interpreted as $((\neg p) \vee q) \rightarrow r$

Semantics

- *Semantics* defines rules for determining the truth of a sentence with respect to a particular model
- The model fixes the truth value of every propositional variable
- Truth values of complex sentences are defined in terms of the truth values of the atomic propositions they contain and the *meaning* of the logical connectives
- Meaning of the logical connectives is given in terms of *truth tables*

Truth Tables

- The truth tables for the four basic connectives are:

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \rightarrow q$
true	*true*	*false*	*true*	*true*	*true*
true	*false*	*false*	*false*	*true*	*false*
false	*true*	*true*	*false*	*true*	*false*
false	*false*	*true*	*false*	*false*	*true*

- Using truth tables we can determine the truth or falsity of any complex sentence in a given model

‘not’ ‘and’ ‘or’ ‘implies’

- The truth tables for $\neg$ and $\wedge$ correspond to how ‘not’ and ‘and’ are used in English
- $p \vee q$ is *true* when p is *true* or q is *true*, or *both* - in English ‘or’ is often taken to be exclusive, e.g., “*I will go to the cinema or I will stay at home*”
- $p \rightarrow q$ is *false* only when p is *true* and q is *false*, and *true* in all other cases - this is not what we usually mean by ‘implication’ in English
  - rather, it says that “if p is *true* then q must be *true*”, so if p is *true* and q is *false*, then the complex sentence is *false*
  - if p is *false* the complex statement is still *true*, whatever the value of q

Example (Truth Tables)

- We can use truth tables to determine the truth or falsity of

$$\neg p \lor q \rightarrow r$$

in the model where *p* is *true*, *q* is *false* and *r* is *false*:

p	q	r	$\neg p$	$\neg p \lor q$	$\neg p \lor q \rightarrow r$
true	*false*	*false*	*false*	*true*	*true*

Example (Truth Tables)

- By enumerating *all* possible models (all possible truth assignments) we can determine the truth value of $\neg p \vee q \rightarrow r$ in all models:

p	q	r	$\neg p$	$\neg p \vee q$	$\neg p \vee q \rightarrow r$
true	*true*	*true*	*false*	*true*	*true*
true	*true*	*false*	*false*	*true*	*false*
true	*false*	*true*	*false*	*false*	*true*
true	*false*	*false*	*false*	*false*	*true*
false	*true*	*true*	*true*	*true*	*true*
false	*true*	*false*	*true*	*true*	*false*
false	*false*	*true*	*true*	*true*	*true*
false	*false*	*false*	*true*	*true*	*false*

Example (More Truth Tables)

- What is the truth table for this formula?

$$(p \rightarrow q) \rightarrow r$$

Logical Equivalence

- Definition
- Two sentences φ and ψ are *logically equivalent* if they are *true* in the same models
  - For example, $p \rightarrow q \equiv \neg p \lor q$:

p	q	$p \rightarrow q$	$\neg p$	$\neg p \lor q$
true	*true*	*true*	*false*	*true*
true	*false*	*false*	*false*	*false*
false	*true*	*true*	*true*	*true*
false	*false*	*true*	*true*	*true*

Satisfiability & Validity

- A sentence is *satisfiable* if it is *true* in some model e.g., $p \lor q$ is *true* in any model in which p is *true* or q is *true*
- A sentence is unsatisfiable if there is no model in which it is *true*, e.g., $p \land \neg p$
- A sentence is valid if it is *true* in all models, e.g., $p \lor \neg p$ is necessarily *true* (tautology)
- A sentence φ is valid if and only if $\neg\varphi$ is unsatisfiable and φ is satisfiable iff $\neg\varphi$ is not valid

Entailment

- Given a notion of truth, we can say what it means for the truth of one statement to follow necessarily from the truth (or falsity) of other statements
- Definition (Entailment)
  - A set of sentences $\{\varphi 1, \varphi 2, \ldots, \varphi n\}$ *entails* a sentence ψ, written $\{\varphi 1, \varphi 2, \ldots, \varphi n\} \models \psi$, if in all models where $\{\varphi 1, \varphi 2, \ldots, \varphi n\}$ are *true*, ψ is also *true*
- Example
  - $\{p \lor q, \neg p\} \models q$ since in all models where $p \lor q$ and $\neg p$ are *true*, q is also *true*
  - Note that $\{p \lor q, p\} \not\models q$ since there is a model where $p \lor q$ and p are *true*, but q is *false*

Entailment

- We can define logical equivalence, validity and satisfiability in terms of entailment:
  - $\varphi \models \psi$ if and only if $\varphi \rightarrow \psi$ is valid (deduction theorem)
  - $\varphi \models \psi$ if and only if $\varphi \land \neg\psi$ is unsatisfiable
  - $\varphi \equiv \psi$ if and only if $\varphi \models \psi$ and $\psi \models \varphi$

Inference

- Entailment can be used to derive conclusions - i.e., to carry out *logical inference*
- By enumerating all possible models we can determine if a sentence ψ follows logically from sentences $\{\varphi 1, \varphi 2, \ldots, \varphi n\}$
- Gives us a reasoning process whose conclusions are guaranteed to be *true* if the premises are *true*
  - if $\{\varphi 1, \varphi 2, \ldots, \varphi n\}$ are *true* in the world, the ψ is necessarily *true* in the world
- Pattern of inference relies only on the truth values of propositions and the meaning of logical connectives, not on detailed knowledge of trains, taxis, meetings etc.

Modus Ponens

- Entailment can be used to show common patterns of inference are valid. For example, the rule of *modus ponens* is:

$$\frac{\varphi, \varphi \rightarrow \psi}{\psi}$$

- We can show that $\varphi, \varphi \rightarrow \psi \models \psi$:

φ	ψ	$\varphi \rightarrow \psi$
true	*true*	*true*
true	*false*	*false*
false	*true*	*true*
false	*false*	*true*

Deduction Theorem

- $\varphi \models \psi$ if and only if $\varphi \rightarrow \psi$ is valid
- $\neg p \wedge (p \vee q) \models q$ if and only if $(\neg p \wedge (p \vee q)) \rightarrow q$ is valid

p	q	$\neg p$	$p \vee q$	$\neg p \wedge (p \vee q)$	$(\neg p \wedge (p \vee q)) \rightarrow q$
true	*true*	*false*	*true*	*false*	*true*
true	*false*	*false*	*true*	*false*	*true*
false	*true*	*true*	*true*	*true*	*true*
false	*false*	*true*	*false*	*false*	*true*

Example (Valid Inference)

- Given the atomic propositions:
  - p: the train is late
  - q: there are taxis at the station
  - r: John is late for the meeting
- Express in propositional calculus:
  1. if the train is late and there are no taxis at the station, John is late for the meeting
  2. the train is late
  3. John is not late for the meeting

  and show using truth tables that there must have been taxis at the station